

A Dirac–Dunkl equation on S^2 and the Bannai–Ito algebra

Hendrik De Bie¹, Vincent X. Genest², and Luc Vinet²

¹Department of Mathematical Analysis, Faculty of Engineering and Architecture, Ghent University, Galglaan 2, 9000 Gent, Belgium

²Centre de recherches mathématiques, Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal, Canada H3C 3J7

Abstract

The Dirac–Dunkl operator on the 2-sphere associated to the \mathbb{Z}_2^3 reflection group is considered. Its symmetries are found and are shown to generate the Bannai–Ito algebra. Representations of the Bannai–Ito algebra are constructed using ladder operators. Eigenfunctions of the spherical Dirac–Dunkl operator are obtained using a Cauchy–Kovalevskaja extension theorem. These eigenfunctions, which correspond to Dunkl monogenics, are seen to support finite-dimensional irreducible representations of the Bannai–Ito algebra.

Keywords: Dirac–Dunkl equation, Bannai–Ito algebra, Cauchy–Kovalevskaja extension

AMS classification numbers: 81Q05, 81R99

1 Introduction

The purpose of this paper is to study the Dirac–Dunkl operator on the two-sphere for the \mathbb{Z}_2^3 reflection group and to investigate its relation with the Bannai–Ito algebra.

The Bannai–Ito algebra is the associative algebra over the field of real numbers with generators I_1 , I_2 , and I_3 satisfying the relations

$$\{I_1, I_2\} = I_3 + \alpha_3, \quad \{I_2, I_3\} = I_1 + \alpha_1, \quad \{I_3, I_1\} = I_2 + \alpha_2, \quad (1)$$

where $\{a, b\} = ab + ba$ is the anticommutator and where α_i , $i = 1, 2, 3$, are real structure constants. The algebra (1) was first presented in [27] as the algebraic structure encoding the bispectral properties of the Bannai–Ito polynomials, which together with the Complementary Bannai–Ito polynomials are the parents of the family of -1 polynomials [15, 27]. The Bannai–Ito algebra also arises in representation theoretic problems [17] and in superintegrable systems [16]; see [2] for a recent overview.

Following their introduction in [8, 9, 10], Dunkl operators have appeared in various areas. They enter the study of Calogero–Moser–Sutherland models [28], they play a central role in the theory of multivariate orthogonal polynomials associated to reflection groups [11], they give rise to families of stochastic processes [20, 25], and they can be used to construct quantum superintegrable systems involving reflections [13, 14]. Dunkl operators also find applications in harmonic analysis and integral transforms [7, 24], as they naturally lead to the Laplace–Dunkl operators, which are second-order differential/difference operator that generalize the standard Laplace operator.

In a recent paper [19], the analysis of the Laplace-Dunkl operator on the two-sphere associated to the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ Abelian reflection group was cast in the frame of the Racah problem for the Hopf algebra $sl_{-1}(2)$ [26], which is closely related to the Lie superalgebra $osp(1|2)$. It was established that the Laplace-Dunkl operator on the two-sphere Δ_{S^2} can be expressed as a quadratic polynomial in the Casimir operator corresponding to the three-fold tensor product of unitary irreducible representations of $sl_{-1}(2)$. A central extension of the Bannai–Ito algebra was seen to emerge as the invariance algebra for Δ_{S^2} and subspaces of the space of Dunkl harmonics that transform according to irreducible representations of the Bannai–Ito algebra were identified.

It is well known that the square root of the standard Laplace operator is the Dirac operator, which is a Clifford-valued first order differential operator. The study of Dirac operators is at the core of Clifford analysis, which can be viewed as a refinement of harmonic analysis [5]. Dirac operators also lend themselves to generalizations involving Dunkl operators [3, 6, 23]. These so-called Dirac-Dunkl operators are the square roots of the corresponding Laplace-Dunkl operators and as such, they exhibit additional structure which makes their analysis both interesting and enlightening.

In this paper, the Dirac-Dunkl operator on the two-sphere associated to the \mathbb{Z}_2^3 Abelian reflection group will be examined. We shall begin by discussing the Laplace- and Dirac- Dunkl operators in \mathbb{R}^3 . The Dirac-Dunkl operator will be defined in terms of the Pauli matrices, which play the role of Dirac’s gamma matrices for the three-dimensional Euclidean space. It will be shown that the Laplace- and Dirac- Dunkl operators can be embedded in a realization of $osp(1|2)$. The notion of Dunkl monogenics, which are homogeneous polynomial null solutions of the Dirac-Dunkl operator, will be reviewed as well as the corresponding Fischer theorem, which describes the decomposition of the space of homogeneous polynomials in terms of Dunkl monogenics. The Dirac-Dunkl operator on the two-sphere, to be called spherical, will then be defined in terms of generalized “angular momentum” operators written in terms of Dunkl operators and its relation with the spherical Laplace-Dunkl operator will be made explicit. The algebraic interpretation of the Dirac-Dunkl operator will proceed from noting its connection with the sCasimir operator of $osp(1|2)$. The symmetries of the spherical Dirac-Dunkl operator will be determined. Remarkably, these symmetries will be seen to satisfy the defining relations of the Bannai–Ito algebra. The relevant finite-dimensional unitary irreducible representations of the Bannai–Ito algebra will be constructed using ladder operators. An explicit basis for the eigenfunctions of the spherical Dirac-Dunkl operator will be obtained. The basis functions, which span the space of Dunkl monogenics, will be constructed systematically using a Cauchy–Kovalevskaia (CK) extension theorem. It will be shown that these spherical wavefunctions, which generalize spherical spinors, transform irreducibly under the action of the Bannai–Ito algebra.

The paper is divided as follows.

- Section 2: Dirac- and Laplace- Dunkl operators in \mathbb{R}^3 and S^2 , $osp(1|2)$ algebra
- Section 3: Symmetries of the Dirac-Dunkl operator on S^2 , Bannai–Ito algebra
- Section 4: Ladder operators, Representations of the Bannai–Ito algebra
- Section 5: CK extension, Eigenfunctions of the spherical Dirac-Dunkl operator

2 Laplace- and Dirac- Dunkl operators for \mathbb{Z}_2^3

In this section, we introduce the Laplace- and Dirac- Dunkl operators associated to the \mathbb{Z}_2^3 reflection group. We show that these operators can be embedded in a realization of $osp(1|2)$. We define the Dunkl monogenics and the Dunkl harmonics and review the Fischer decomposition theorem. We introduce the spherical Laplace- and Dirac- Dunkl operators and we give their relation.

2.1 Laplace– and Dirac– Dunkl operators in \mathbb{R}^3

Let $\vec{x} = (x_1, x_2, x_3)$ denote the coordinate vector in \mathbb{R}^3 and let μ_i , $i = 1, 2, 3$, be real numbers such that $\mu_i \geq 0$. The Dunkl operators associated to the \mathbb{Z}_2^3 reflection group, denoted by T_i , are given by

$$T_i = \partial_{x_i} + \frac{\mu_i}{x_i}(1 - R_i), \quad i = 1, 2, 3, \quad (2)$$

where

$$R_i f(x_i) = f(-x_i)$$

is the reflection operator. It is obvious that the operators T_i , T_j commute with one another. We define the Dirac–Dunkl operator in \mathbb{R}^3 , to be denoted by \underline{D} , as follows:

$$\underline{D} = \sigma_1 T_1 + \sigma_2 T_2 + \sigma_3 T_3, \quad (3)$$

where the σ_i are the familiar Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These matrices satisfy the identities

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij},$$

where $[a, b] = ab - ba$ is the commutator, where ϵ_{ijk} is the Levi-Civita symbol and where summation over repeated indices is implied. The Pauli matrices provide a representation of the Euclidean Clifford algebra with three generators on \mathbb{C}^2 , i.e. on the space of two-spinors. Indeed, one has

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}, \quad i, j = 1, 2, 3. \quad (4)$$

As a direct consequence of (4), one has

$$\underline{D}^2 = \Delta = T_1^2 + T_2^2 + T_3^2, \quad (5)$$

where Δ is the Laplace–Dunkl operator in \mathbb{R}^3 .

The Dirac–Dunkl and the Laplace–Dunkl operators (3) and (5) can be embedded in a realization of the Lie superalgebra $\mathfrak{osp}(1|2)$. Let \underline{x} and $||\vec{x}||^2$ be the operators defined by

$$\underline{x} = \sigma_1 x_1 + \sigma_2 x_2 + \sigma_3 x_3, \quad ||\vec{x}||^2 = \underline{x}^2 = x_1^2 + x_2^2 + x_3^2,$$

and let \mathbb{E} stand for the Euler (or dilation) operator

$$\mathbb{E} = x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3}.$$

A direct calculation shows that one has

$$\begin{aligned} \{\underline{x}, \underline{x}\} &= 2||\vec{x}||^2, & \{\underline{D}, \underline{D}\} &= 2\Delta, & \{\underline{x}, \underline{D}\} &= 2(\mathbb{E} + \gamma_3), \\ [\underline{D}, ||\vec{x}||^2] &= 2\underline{x}, & [\mathbb{E} + \gamma_3, \underline{x}] &= \underline{x}, & [\mathbb{E} + \gamma_3, \underline{D}] &= -\underline{D}, & [\Delta, \underline{x}] &= 2\underline{D}, \\ [\mathbb{E} + \gamma_3, \Delta] &= -2\Delta, & [\mathbb{E} + \gamma_3, ||\vec{x}||^2] &= 2||\vec{x}||^2, & [\Delta, ||\vec{x}||^2] &= 4(\mathbb{E} + \gamma_3), \end{aligned} \quad (6)$$

where

$$\gamma_3 = \mu_1 + \mu_2 + \mu_3 + 3/2.$$

The commutation relations (6) are seen to correspond to those of the $\mathfrak{osp}(1|2)$ Lie superalgebra [12]. In fact, the relations (6) hold in any dimension and for any choice of the reflection group with different values of the constant γ [3, 23].

Let $\mathcal{P}_N(\mathbb{R}^3)$ denote the space of homogeneous polynomials of degree N in \mathbb{R}^3 , where N is a non-negative integer. The space of *Dunkl monogenics* of degree N for the reflection group \mathbb{Z}_2^3 shall be denoted by $\mathcal{M}_N(\mathbb{R}^3)$. It is defined as

$$\mathcal{M}_N(\mathbb{R}^3) := \text{Ker } \underline{D} \cap (\mathcal{P}_N(\mathbb{R}^3) \otimes \mathbb{C}^2). \quad (7)$$

Similarly, the space of scalar *Dunkl harmonics* of degree N for the reflection group \mathbb{Z}_2^3 is denoted by $\mathcal{H}_N(\mathbb{R}^3)$ and defined as [11]

$$\mathcal{H}_N(\mathbb{R}^3) := \text{Ker } \Delta \cap \mathcal{P}_N(\mathbb{R}^3).$$

The space of spinor-valued Dunkl harmonics has a direct sum decomposition in terms of the Dunkl monogenics. This decomposition reads

$$\mathcal{H}_N(\mathbb{R}^3) \otimes \mathbb{C}^2 = \mathcal{M}_N(\mathbb{R}^3) \oplus \underline{x} \mathcal{M}_{N-1}(\mathbb{R}^3).$$

For $\gamma_3 > 0$, which is automatically satisfied when $\mu_1, \mu_2, \mu_3 \geq 0$, the following direct sum decomposition holds [23]:

$$\mathcal{P}_N(\mathbb{R}^3) \otimes \mathbb{C}^2 = \bigoplus_{k=0}^N \underline{x}^k \mathcal{M}_{N-k}(\mathbb{R}^3). \quad (8)$$

The above is called the *Fischer decomposition* and will play an important role in what follows.

2.2 Laplace- and Dirac- operators on S^2

The explicit expression for the Dunkl operators (2) allows to write the Laplace–Dunkl operator (5) as

$$\Delta = \sum_{i=1}^3 \partial_{x_i}^2 + \frac{2\mu_i}{x_i} \partial_{x_i} - \frac{\mu_i}{x_i^2} (1 - R_i). \quad (9)$$

Since reflections are elements of $O(3)$, the Laplace–Dunkl operator (9), like the standard Laplace operator, separates in spherical coordinates. Consequently, it can be restricted to functions defined on the unit sphere. Let Δ_{S^2} denote the restriction of (9) to the two-sphere, which shall be referred to as the spherical Laplace–Dunkl operator. It is seen that Δ_{S^2} can be expressed as

$$\Delta_{S^2} = \|\vec{x}\|^2 \Delta - \mathbb{E}(\mathbb{E} + 2\mu_1 + 2\mu_2 + 2\mu_3 + 1). \quad (10)$$

The spherical Laplace–Dunkl operator can also be written in terms of the Dunkl angular momentum operators. These operators are defined as

$$L_1 = \frac{1}{i}(x_2 T_3 - x_3 T_2), \quad L_2 = \frac{1}{i}(x_3 T_1 - x_1 T_3), \quad L_3 = \frac{1}{i}(x_1 T_2 - x_2 T_1), \quad (11)$$

and satisfy the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k (1 + 2\mu_k R_k), \quad [L_i, R_i] = 0, \quad \{L_i, R_j\} = 0. \quad (12)$$

Taking into account the relation (10), a direct calculation shows that [18]

$$-\Delta_{S^2} = L_1^2 + L_2^2 + L_3^2 - 2 \sum_{1 \leq i < j \leq 3} \mu_i \mu_j (1 - R_i R_j) - \sum_{1 \leq j \leq 3} \mu_j (1 - R_j). \quad (13)$$

When $\mu_1 = \mu_2 = \mu_3 = 0$, the relation (13) reduces to the standard relation between the Laplace operator on the two-sphere and the angular momentum operators.

Let us now introduce the main object of study: the Dirac–Dunkl operator on the two-sphere. This operator, denoted by Γ , is defined as

$$\Gamma = \vec{\sigma} \cdot \vec{L} + \vec{\mu} \cdot \vec{R}, \quad (14)$$

where $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ and $\vec{R} = (R_1, R_2, R_3)$. When $\mu_1 = \mu_2 = \mu_3 = 0$, all reflections disappear and (14) reduces to the standard Hamiltonian describing spin-orbit interaction. The operator (14) is linked to the spherical Laplace–Dunkl operator by a quadratic relation. Upon using the equations (4), (12) and (13), one finds that

$$\Gamma^2 + \Gamma = -\Delta_{S^2} + (\mu_1 + \mu_2 + \mu_3)(\mu_1 + \mu_2 + \mu_3 + 1). \quad (15)$$

A relation akin to (15) was derived in [19]; it involved a scalar operator instead of Γ . The Dirac–Dunkl operator on the two-sphere has a natural algebraic interpretation in terms of the realization (6) of the $\mathfrak{osp}(1|2)$ algebra. It corresponds, up to an additive constant, to the so-called sCasimir operator. Indeed, it is verified that

$$\{\Gamma + 1, \underline{x}\} = 0, \quad \{\Gamma + 1, \underline{D}\} = 0,$$

and that

$$[\Gamma + 1, \mathbb{E}] = 0, \quad [\Gamma + 1, \|\vec{x}\|^2] = 0, \quad [\Gamma + 1, \Delta] = 0.$$

Hence $\Gamma + 1$ anticommutes with the odd generators and commutes with the even generators of $\mathfrak{osp}(1|2)$, which is the defining property of the sCasimir operator [22]. The spherical Dirac–Dunkl operator is usually written as a commutator (see for example [4]). For (14), one has

$$\Gamma + 1 = \frac{1}{2}([\underline{D}, \underline{x}] - 1). \quad (16)$$

The space of Dunkl monogenics $\mathcal{M}_N(\mathbb{R}^3)$ of degree N is an eigenspace for this operator. Indeed, upon using (16), the $\mathfrak{osp}(1|2)$ relations (6) and the fact that

$$\underline{D} \mathcal{M}_N(\mathbb{R}^3) = 0, \quad \mathbb{E} \mathcal{M}_N(\mathbb{R}^3) = N \mathcal{M}_N(\mathbb{R}^3),$$

one can write

$$\begin{aligned} (\Gamma + 1) \mathcal{M}_N(\mathbb{R}^3) &= \frac{1}{2}([\underline{D}, \underline{x}] - 1) \mathcal{M}_N(\mathbb{R}^3) = \frac{1}{2}(\underline{D} \underline{x} - 1) \mathcal{M}_N(\mathbb{R}^3) \\ &= \frac{1}{2}(\{\underline{x}, \underline{D}\} - 1) \mathcal{M}_N(\mathbb{R}^3) = \frac{1}{2}(2(\mathbb{E} + \gamma_3) - 1) \mathcal{M}_N(\mathbb{R}^3), \end{aligned}$$

which gives

$$(\Gamma + 1) \mathcal{M}_N(\mathbb{R}^3) = (N + \mu_1 + \mu_2 + \mu_3 + 1) \mathcal{M}_N(\mathbb{R}^3), \quad (17)$$

where $N = 0, 1, 2, \dots$ is a non-negative integer.

3 Symmetries of the spherical Dirac–Dunkl operator

In this section, the symmetries of the spherical Dirac–Dunkl operator are obtained and are seen to satisfy the defining relations of the Bannai–Ito algebra.

Introduce the operators J_i defined by

$$J_i = L_i + \sigma_i(\mu_j R_j + \mu_k R_k + 1/2), \quad i = 1, 2, 3, \quad (18)$$

where (ijk) is a cyclic permutation of $\{1, 2, 3\}$. The operators J_i are symmetries of the spherical Dirac–Dunkl operator, as it is verified that

$$[\Gamma, J_i] = 0, \quad i = 1, 2, 3.$$

The operator Γ can be expressed in terms of the symmetries J_i in the following way:

$$\Gamma = \sigma_1 J_1 + \sigma_2 J_2 + \sigma_3 J_3 - \mu_1 R_1 - \mu_2 R_2 - \mu_3 R_3 - 3/2.$$

A direct calculation shows that the operators J_i satisfy the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk} \left(J_k + 2\mu_k (\Gamma + 1) \sigma_k R_k + 2\mu_i \mu_j \sigma_k R_i R_j \right). \quad (19)$$

The operator Γ also admits the three involutions

$$Z_i = \sigma_i R_i, \quad Z_i^2 = 1, \quad i = 1, 2, 3, \quad (20)$$

as symmetry operators, i.e

$$[\Gamma, Z_i] = 0, \quad i = 1, 2, 3.$$

The commutation relations between Z_i and J_i read

$$[J_i, Z_i] = 0, \quad \{J_i, Z_j\} = 0, \quad \{Z_i, Z_j\} = 0, \quad i \neq j. \quad (21)$$

The involutions (20) and the relations (21) can be exploited to give another presentation of the symmetry algebra of Γ . Let K_i , $i = 1, 2, 3$, be defined as follows

$$K_i = -i J_i Z_j Z_k, \quad (22)$$

where (ijk) is again a cyclic permutation of $\{1, 2, 3\}$. Since the operators J_i and Z_i both commute with Γ , it follows that the operators K_i also commute with Γ . Upon combining the relations (19) and (21), one finds that the symmetries K_i satisfy the commutation relations

$$\{K_i, K_j\} = \epsilon_{ijk} \left(K_k + 2\mu_k (\Gamma + 1) R_1 R_2 R_3 + 2\mu_i \mu_j \right). \quad (23)$$

The invariance algebra (23) of the Dirac–Dunkl operator thus has the form of the Bannai–Ito algebra. More precisely, (23) can be viewed as a central extension of the Bannai–Ito algebra given the presence of the central element $(\Gamma + 1) R_1 R_2 R_3$ on the right hand side. In terms of the symmetries K_i , the Γ operator reads

$$\Gamma = K_1 R_2 R_3 + K_2 R_1 R_3 + K_3 R_1 R_2 - \mu_1 R_1 - \mu_2 R_2 - \mu_3 R_3 - 3/2. \quad (24)$$

The commutation relations between the symmetries K_i and the involutions Z_i are

$$[K_i, Z_j] = 0.$$

The Casimir operator of the Bannai–Ito algebra, denoted by Q , has the expression [27]

$$Q = K_1^2 + K_2^2 + K_3^2. \quad (25)$$

It is easily verified that Q commutes with K_i and Z_i for $i = 1, 2, 3$. A direct calculation shows that in the realization (22), the Casimir operator (25) can be written as

$$Q = (\Gamma + 1)^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 - 1/4.$$

It follows from (17) and (23) that the space of Dunkl monogenics $\mathcal{M}_N(\mathbb{R}^3)$ of degree N carries representations of the Bannai–Ito algebra (1). The precise content of $\mathcal{M}_N(\mathbb{R}^3)$ in representations of the Bannai–Ito algebra will be determined in section 5.

4 Representations of the Bannai–Ito algebra

In this section, the representations of the Bannai–Ito algebra corresponding to the realization (23) are constructed using ladder operators.

On the space of Dunkl monogenics $\mathcal{M}_N(\mathbb{R}^3)$ of degree N , the symmetries K_i of the Dirac–Dunkl operator satisfy the commutation relations

$$\{K_1, K_2\} = K_3 + \omega_3, \quad \{K_2, K_3\} = K_1 + \omega_1, \quad \{K_3, K_1\} = K_2 + \omega_2, \quad (26)$$

with structure constants

$$\omega_3 = 2\mu_1\mu_2 + 2\mu_3\mu_N, \quad \omega_1 = 2\mu_2\mu_3 + 2\mu_1\mu_N, \quad \omega_2 = 2\mu_3\mu_1 + 2\mu_2\mu_N, \quad (27)$$

and where we have defined

$$\mu_N = (-1)^N(N + \mu_1 + \mu_2 + \mu_3 + 1). \quad (28)$$

On $\mathcal{M}_N(\mathbb{R}^3)$, the Casimir operator (25) takes the value

$$Q = (N + \mu_1 + \mu_2 + \mu_3 + 1)^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 - 1/4 \equiv q_N. \quad (29)$$

We seek to construct the representations of the Bannai–Ito algebra (26) on the space spanned by the orthonormal basis vectors $|N, k\rangle$ characterized by the eigenvalue equations

$$K_3 |N, k\rangle = \lambda_k |N, k\rangle, \quad Q |N, k\rangle = q_N |N, k\rangle. \quad (30)$$

Since the operators K_i are potential observables, it is formally assumed that

$$K_i^\dagger = K_i, \quad i = 1, 2, 3. \quad (31)$$

To characterize the representation, one needs to determine the spectrum of K_3 and the action of the operator K_1 on the basis vectors $|N, k\rangle$.

Introduce the operators K_+ and K_- defined by [27]

$$\begin{aligned} K_+ &= (K_1 + K_2)(K_3 - 1/2) - (\omega_1 + \omega_2)/2, \\ K_- &= (K_1 - K_2)(K_3 + 1/2) + (\omega_1 - \omega_2)/2. \end{aligned} \quad (32)$$

Using the defining relations (26) and the Hermiticity condition (31), it is seen that the operators K_{\pm} are skew-Hermitian, i.e.

$$K_{\pm}^{\dagger} = -K_{\pm}.$$

Moreover, a direct calculation shows that they satisfy the commutation relations

$$\{K_3, K_+\} = K_+, \quad \{K_3, K_-\} = -K_-, \quad (33)$$

whence it follows that

$$[K_3, K_+^2] = 0, \quad [K_3, K_-^2] = 0.$$

Using (33), one can write

$$\begin{aligned} K_3 K_+ |N, k\rangle &= (K_+ - K_+ K_3) |N, k\rangle = (1 - \lambda_k) K_+ |N, k\rangle, \\ K_3 K_- |N, k\rangle &= (-K_- - K_- K_3) |N, k\rangle = (-1 - \lambda_k) K_- |N, k\rangle. \end{aligned} \quad (34)$$

The above relations indicate that $K_+ |N, k\rangle$ and $K_- |N, k\rangle$ are eigenvectors of K_3 with eigenvalues $(1 - \lambda_k)$ and $-(1 + \lambda_k)$, respectively. One has the two inequalities

$$\|K_+ |N, k\rangle\|^2 = \langle N, k | K_+^{\dagger} K_+ |N, k\rangle \geq 0, \quad (35a)$$

$$\|K_- |N, k\rangle\|^2 = \langle N, k | K_-^{\dagger} K_- |N, k\rangle \geq 0. \quad (35b)$$

Consider the LHS of (35a). The operator K_+^{\dagger} is of the form

$$K_+^{\dagger} = (K_3 - 1/2)(K_1 + K_2) - (\omega_1 + \omega_2)/2.$$

Upon using the commutation relations (26) and (29), it is seen that

$$K_+^{\dagger} K_+ = (K_3 - 1/2)^2 (Q - K_3^2 + K_3 + \omega_3) - (\omega_1 + \omega_2)^2/4. \quad (36)$$

Using the above expression in (35a) with (27) and (29), one finds a factorized form for the inequality

$$-(\mu_1 + \mu_2 + 1/2 - \lambda_k)(\mu_N + \mu_3 + 1/2 - \lambda_k)(\lambda_k + \mu_1 + \mu_2 - 1/2)(\lambda_k + \mu_N + \mu_3 - 1/2) \geq 0. \quad (37)$$

which is equivalent to

$$\begin{cases} \mu_1 + \mu_2 \leq |\lambda_k - 1/2| \leq N + \mu_1 + \mu_2 + 2\mu_3 + 1, & N \text{ even,} \\ \mu_1 + \mu_2 \leq |\lambda_k - 1/2| \leq N + \mu_1 + \mu_2 + 1, & N \text{ odd.} \end{cases} \quad (38)$$

Proceeding similarly for K_- , one can write

$$K_-^{\dagger} K_- = (K_3 + 1/2)^2 (Q - K_3^2 - K_3 - \omega_3) - (\omega_1 - \omega_2)^2/4. \quad (39)$$

and one finds that (35b) amounts to

$$-(\mu_1 - \mu_2 - 1/2 - \lambda_k)(\mu_3 - \mu_N - 1/2 - \lambda_k)(\lambda_k + \mu_1 - \mu_2 + 1/2)(\lambda_k + \mu_3 - \mu_N + 1/2) \geq 0, \quad (40)$$

which can be expressed as

$$\begin{cases} |\mu_1 - \mu_2| \leq |\lambda_k + 1/2| \leq N + \mu_1 + \mu_2 + 1, & N \text{ even,} \\ |\mu_1 - \mu_2| \leq |\lambda_k + 1/2| \leq N + \mu_1 + \mu_2 + 2\mu_3 + 1, & N \text{ odd,} \end{cases} \quad (41)$$

Upon combining (38) and (41), we choose

$$\lambda_0 = \mu_1 + \mu_2 + 1/2,$$

whence it follows from (37) that

$$K_+|N, 0\rangle = 0.$$

Let us mention that the other choice $\tilde{\lambda}_0 = -\mu_1 - \mu_2 + 1/2$ permitted by (38) only leads to trivial representations.

Starting from the vector $|N, 0\rangle$ with eigenvalue λ_0 , one can obtain a string of eigenvectors of K_3 with different eigenvalues by successively applying K_+ and K_- . The eigenvalues

$$\lambda_k = (-1)^k(k + \mu_1 + \mu_2 + 1/2), \quad k = 0, 1, 2, 3, \dots \quad (42)$$

are obtained by applying K_3 on the vectors

$$|N, 0\rangle, \quad K_-|N, 0\rangle, \quad K_+K_-|N, 0\rangle, \quad K_-K_+K_-|N, 0\rangle, \dots \quad (43)$$

One needs to alternate the application of K_+ and K_- since K_\pm^2 commute with K_3 and hence their action does not produce an eigenvector with a different eigenvalue. Using (42), one can write

$$\|K_+|N, k\rangle\|^2 = \begin{cases} \rho_k^{(N)}, & k \text{ even}, \\ \rho_{k+1}^{(N)}, & k \text{ odd}, \end{cases}$$

where

$$\rho_k^{(N)} = -k(k + 2\mu_1 + 2\mu_2)(k + \mu_1 + \mu_2 + \mu_3 + \mu_N)(k + \mu_1 + \mu_2 - \mu_3 - \mu_N), \quad (44)$$

and also

$$\|K_-|N, k\rangle\|^2 = \begin{cases} \sigma_{k+1}^{(N)}, & k \text{ even}, \\ \sigma_k^{(N)}, & k \text{ odd}, \end{cases}$$

with

$$\sigma_k^{(N)} = -(k + 2\mu_1)(k + 2\mu_2)(k + \mu_1 + \mu_2 - \mu_3 + \mu_N)(k + \mu_1 + \mu_2 + \mu_3 - \mu_N). \quad (45)$$

It is verified that the positivity conditions $\rho_k^{(N)} > 0$ and $\sigma_k^{(N)} > 0$ are satisfied for all $k = 0, 1, \dots, N$, provided that $\mu_i \geq 0$ for $i = 1, 2, 3$. Following (43), (44) and (45), we define the orthonormal basis vectors $|N, k\rangle$ from $|N, 0\rangle$ as follows:

$$|N, k+1\rangle = \begin{cases} \frac{1}{\sqrt{\|K_-|N, k\rangle\|^2}} K_-|N, k\rangle, & k \text{ even}, \\ \frac{-1}{\sqrt{\|K_+|N, k\rangle\|^2}} K_+|N, k\rangle, & k \text{ odd}, \end{cases} \quad (46)$$

where the phase factor was chosen to ensure the condition $K_\pm^\dagger = -K_\pm$. From (36), (39), (43) and (46), the actions of the ladder operators K_\pm are seen to have the expressions

$$\begin{aligned} K_+|N, k\rangle &= \begin{cases} \sqrt{\rho_k^{(N)}} |N, k-1\rangle, & k \text{ even}, \\ -\sqrt{\rho_{k+1}^{(N)}} |N, k+1\rangle, & k \text{ odd}, \end{cases} \\ K_-|N, k\rangle &= \begin{cases} \sqrt{\sigma_{k+1}^{(N)}} |N, k+1\rangle, & k \text{ even}, \\ -\sqrt{\sigma_k^{(N)}} |N, k-1\rangle, & k \text{ odd}. \end{cases} \end{aligned} \quad (47)$$

As is observed in (44) and (45), one has $K_+|N, N\rangle = 0$ when N is odd and $K_-|N, N\rangle = 0$ when N is even. As a result, the representation has dimension $N + 1$. Moreover, it immediately follows from the actions (47) that the representation is irreducible, as there are no invariant subspaces.

Let us now give the actions of the generators. The eigenvalues of K_3 are of the form

$$K_3|N, k\rangle = (-1)^k(k + \mu_1 + \mu_2 + 1/2)|N, k\rangle, \quad k = 0, 1, \dots, N.$$

The action of the operator K_1 in the basis $|N, k\rangle$ can be obtained directly from the definitions (32) and the actions (47). One finds that K_1 acts in the tridiagonal fashion

$$K_1|N, k\rangle = U_{k+1}|N, k+1\rangle + V_k|N, k\rangle + U_k|N, k-1\rangle,$$

with

$$U_k = \sqrt{A_{k-1}C_k}, \quad V_k = \mu_2 + \mu_3 + 1/2 - A_k - C_k,$$

where the coefficients A_k and C_k read

$$\begin{aligned} A_k &= \begin{cases} \frac{(k+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3-\mu_N+1)}{2(k+\mu_1+\mu_2+1)}, & k \text{ even}, \\ \frac{(k+2\mu_1+2\mu_2+1)(k+\mu_1+\mu_2+\mu_3+\mu_N+1)}{2(k+\mu_1+\mu_2+1)}, & k \text{ odd}, \end{cases} \\ C_k &= \begin{cases} -\frac{k(k+\mu_1+\mu_2-\mu_3-\mu_N)}{2(k+\mu_1+\mu_2)}, & k \text{ even}, \\ -\frac{(k+2\mu_1)(k+\mu_1+\mu_2-\mu_3+\mu_N)}{2(k+\mu_1+\mu_2)}, & k \text{ odd}. \end{cases} \end{aligned} \quad (48)$$

For $\mu_i \geq 0$, $i = 1, 2, 3$, one has $U_\ell > 0$ for $\ell = 1, \dots, N$ and $U_0 = U_{N+1} = 0$. Hence in the basis $|N, k\rangle$, the operator K_1 is represented by a symmetric $(N + 1) \times (N + 1)$ matrix.

It is observed that the commutation relations (26) along with the structure constants (27) and the Casimir value (29) are invariant under any cyclic permutation of the pairs (K_i, μ_i) for $i = 1, 2, 3$. Consequently, the matrix elements of the generators in other bases, for example bases in which K_1 or K_2 are diagonal, can be obtained directly by applying the corresponding cyclic permutation on the parameters μ_i .

5 Eigenfunctions of the spherical Dirac–Dunkl operator

In this section, a basis for the space of Dunkl monogenics $\mathcal{M}_N(\mathbb{R}^3)$ of degree N is constructed using a Cauchy-Kovalevskaja extension theorem. It is shown that the basis functions transform irreducibly under the action of the Bannai–Ito algebra. The wavefunctions are shown to be orthogonal with respect to a scalar product defined as an integral over the 2-sphere.

5.1 Cauchy-Kovalevskaja map

Let \tilde{D} , \tilde{x} and $\tilde{\mathbb{E}}$ be defined as follows:

$$\tilde{D} = \sigma_1 T_1 + \sigma_2 T_2, \quad \tilde{x} = \sigma_1 x_1 + \sigma_2 x_2, \quad \tilde{\mathbb{E}} = x_1 \partial_{x_1} + x_2 \partial_{x_2}.$$

There is an isomorphism $\mathbf{CK}_{x_3}^{\mu_3} : \mathcal{P}_N(\mathbb{R}^2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{M}_N(\mathbb{R}^3)$, between the space of spinor-valued homogeneous polynomials of degree N in the variables (x_1, x_2) and the space of Dunkl monogenics of degree N in the variables (x_1, x_2, x_3) .

Proposition 1. *The isomorphism $\mathbf{CK}_{x_3}^{\mu_3}$ between $\mathcal{P}_N(\mathbb{R}^2) \otimes \mathbb{C}^2$ and $\mathcal{M}_N(\mathbb{R}^3)$ has the explicit expression*

$$\mathbf{CK}_{x_3}^{\mu_3} = {}_0F_1 \left(\begin{matrix} - \\ \mu_3 + 1/2 \end{matrix} \middle| - \left(\frac{x_3 \tilde{D}}{2} \right)^2 \right) - \frac{\sigma_3 x_3 \tilde{D}}{2\mu_3 + 1} {}_0F_1 \left(\begin{matrix} - \\ \mu_3 + 3/2 \end{matrix} \middle| - \left(\frac{x_3 \tilde{D}}{2} \right)^2 \right), \quad (49)$$

where ${}_pF_q$ is the generalized hypergeometric series [1].

Proof. Let $p(x_1, x_2) \in \mathcal{P}_n(\mathbb{R}^2) \otimes \mathbb{C}^2$. We set

$$\mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)] = \sum_{\alpha=0}^n (\sigma_3 x_3)^\alpha p_\alpha(x_1, x_2),$$

with $p_0(x_1, x_2) \equiv p(x_1, x_2)$ and $p_\alpha(x_1, x_2) \in \mathcal{P}_{n-\alpha}(\mathbb{R}^2) \otimes \mathbb{C}^2$ and we determine the $p_\alpha(x_1, x_2)$ such that $\mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)]$ is the kernel of \underline{D} . One has

$$\begin{aligned} \underline{D} \mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)] &= \sum_{\alpha=0}^n (-\sigma_3 x_3)^\alpha (\sigma_1 T_1 + \sigma_2 T_2) p_\alpha(x_1, x_2) + \sum_{\alpha=1}^n \sigma_3^{\alpha+1} (T_3 x_3^\alpha) p_\alpha(x_1, x_2) \\ &= \sum_{\alpha=0}^n (-\sigma_3 x_3)^\alpha (\sigma_1 T_1 + \sigma_2 T_2) p_\alpha(x_1, x_2) + \sum_{\alpha=1}^n \sigma_3^{\alpha+1} [\alpha + \mu_3(1 - (-1)^\alpha)] x_3^{\alpha-1} p_\alpha(x_1, x_2). \end{aligned}$$

Imposing the condition $\underline{D} \mathbf{CK}_{x_3}^{\mu_3}[p(x_1, x_2)] = 0$ leads to the equations

$$\sum_{\alpha=0}^n (-1)^{\alpha+1} (\sigma_3 x_3)^\alpha (\sigma_1 T_1 + \sigma_2 T_2) p_\alpha(x_1, x_2) = \sum_{\alpha=0}^{n-1} (\sigma_3 x_3)^\alpha [\alpha + \mu_3(1 + (-1)^\alpha)] p_{\alpha+1}(x_1, x_2),$$

from which one finds that

$$\begin{aligned} p_{2\alpha}(x_1, x_2) &= \left[\frac{(-1)^\alpha}{2^{2\alpha} \alpha! (\mu_3 + 1/2)_\alpha} \right] (\sigma_1 T_1 + \sigma_2 T_2)^{2\alpha} p(x_1, x_2), \\ p_{2\alpha+1}(x_1, x_2) &= \left[\frac{(-1)^{\alpha+1}}{2^{2\alpha+1} \alpha! (\mu_3 + 1/2)(\mu_3 + 3/2)_\alpha} \right] (\sigma_1 T_1 + \sigma_2 T_2)^{2\alpha+1} p(x_1, x_2), \end{aligned}$$

where $(a)_n$ stands for the Pochhammer symbol. It is seen that the above corresponds to the hypergeometric expression (49). \square

The inverse of the isomorphism $\mathbf{CK}_{x_3}^{\mu_3}$ is clearly given by I_{x_3} with $I_{x_3} f(x_1, x_2, x_3) = f(x_1, x_2, 0)$. When $\mu_3 = 0$, the operator $\mathbf{CK}_{x_3}^{\mu_3}$ reduces to the well-known Cauchy-Kovalevskaja extension operator for the standard Dirac operator, as determined in [5]. It is manifest that proposition 1 can be extended to any dimension. Thus, in a similar fashion, one has the isomorphism

$$\mathbf{CK}_{x_2}^{\mu_2} : \mathcal{P}_k(\mathbb{R}) \otimes \mathbb{C}^2 \longrightarrow \mathcal{M}_k(\mathbb{R}^2),$$

between the space of spinor-valued homogeneous polynomials in the variable x_1 and the space of Dunkl monogenics of degree k in the variables (x_1, x_2) . This isomorphism has the explicit expression

$$\mathbf{CK}_{x_2}^{\mu_2} = {}_0F_1 \left(\begin{matrix} - \\ \mu_2 + 1/2 \end{matrix} \middle| - \left(\frac{x_2 \sigma_1 T_1}{2} \right)^2 \right) - \frac{\sigma_2 x_2 (\sigma_1 T_1)}{2\mu_2 + 1} {}_0F_1 \left(\begin{matrix} - \\ \mu_2 + 3/2 \end{matrix} \middle| - \left(\frac{x_2 \sigma_1 T_1}{2} \right)^2 \right). \quad (50)$$

5.2 A basis for $\mathcal{M}_N(\mathbb{R}^3)$

Let us now show how a basis for the space of Dunkl monogenics of degree N in \mathbb{R}^3 can be constructed using the $\mathbf{CK}_{x_i}^{\mu_i}$ extension operators and the Fischer decomposition theorem (8). Let $\chi_+ = (1, 0)^\top$ and $\chi_- = (0, 1)^\top$ denote the basis spinors; one has $\mathbb{C}^2 = \text{Span}\{\chi_\pm\}$. Consider the following tower of \mathbf{CK} extensions and Fischer decompositions:

$$\begin{array}{ccc}
 & & \mathcal{P}_N(\mathbb{R}^2) \otimes \mathbb{C}^2 \xrightarrow{\mathbf{CK}_{x_3}^{\mu_3}} \mathcal{M}_N(\mathbb{R}^3) \\
 & & \parallel \\
 \text{Span}\{x_1^N \chi_\pm\} = \mathcal{P}_N(\mathbb{R}) \otimes \mathbb{C}^2 & \xrightarrow{\mathbf{CK}_{x_2}^{\mu_2}} & \mathcal{M}_N(\mathbb{R}^2) \rightsquigarrow \mathcal{M}_N(\mathbb{R}^2) \\
 & & \oplus \\
 \text{Span}\{x_1^{N-1} \chi_\pm\} = \mathcal{P}_{N-1}(\mathbb{R}) \otimes \mathbb{C}^2 & \xrightarrow{\mathbf{CK}_{x_2}^{\mu_2}} & \mathcal{M}_{N-1}(\mathbb{R}^2) \rightsquigarrow \tilde{x} \mathcal{M}_{N-1}(\mathbb{R}^2) \\
 & & \oplus \\
 \vdots & & \vdots \\
 & & \oplus \\
 \text{Span}\{x_1^k \chi_\pm\} = \mathcal{P}_k(\mathbb{R}) \otimes \mathbb{C}^2 & \xrightarrow{\mathbf{CK}_{x_2}^{\mu_2}} & \mathcal{M}_k(\mathbb{R}^2) \rightsquigarrow \tilde{x}^{N-k} \mathcal{M}_k(\mathbb{R}^2) \quad \sim \psi_{k,\pm}^{(N)} \\
 & & \oplus \\
 \vdots & & \vdots \\
 & & \oplus \\
 \text{Span}\{x_1 \chi_\pm\} = \mathcal{P}_1(\mathbb{R}) \otimes \mathbb{C}^2 & \xrightarrow{\mathbf{CK}_{x_2}^{\mu_2}} & \mathcal{M}_1(\mathbb{R}^2) \rightsquigarrow \tilde{x}^{N-1} \mathcal{M}_1(\mathbb{R}^2) \\
 & & \oplus \\
 \text{Span}\{\chi_\pm\} = \mathcal{P}_0(\mathbb{R}) \otimes \mathbb{C}^2 & \xrightarrow{\mathbf{CK}_{x_2}^{\mu_2}} & \mathcal{M}_0(\mathbb{R}^2) \rightsquigarrow \tilde{x}^N \mathcal{M}_0(\mathbb{R}^2)
 \end{array}$$

Diagram 1. Horizontally, application of the \mathbf{CK} map and multiplication by \tilde{x} . Vertically, Fischer decomposition theorem for $\mathcal{P}_N(\mathbb{R}^2) \otimes \mathbb{C}^2$.

As can be seen from the above diagram, the spinors

$$\psi_{k,\pm}^{(N)} = \mathbf{CK}_{x_3}^{\mu_3} \left[\tilde{x}^{N-k} \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] \right] \chi_\pm, \quad k = 0, 1, \dots, N, \quad (51)$$

provide a basis for the space of Dunkl monogenics of degree N in (x_1, x_2, x_3) . The basis spinors (51) can be calculated explicitly. To perform the calculation, one needs the identities

$$\begin{aligned}
 \tilde{D}^{2\alpha} \tilde{x}^{2\beta} M_k &= 2^{2\alpha} (-\beta)_\alpha (1 - k - \beta - \gamma_2)_\alpha \tilde{x}^{2\beta-2\alpha} M_k, \\
 \tilde{D}^{2\alpha+1} \tilde{x}^{2\beta} M_k &= \beta 2^{2\alpha+1} (1 - \beta)_\alpha (1 - k - \beta - \gamma_2)_\alpha \tilde{x}^{2\beta-2\alpha-1} M_k, \\
 \tilde{D}^{2\alpha} \tilde{x}^{2\beta+1} M_k &= 2^{2\alpha} (-\beta)_\alpha (-k - \beta - \gamma_2)_\alpha \tilde{x}^{2\beta-2\alpha+1} M_k, \\
 \tilde{D}^{2\alpha+1} \tilde{x}^{2\beta+1} M_k &= (k + \beta + \gamma_2) 2^{2\alpha+1} (-\beta)_\alpha (1 - k - \beta - \gamma_2)_\alpha \tilde{x}^{2\beta-2\alpha} M_k,
 \end{aligned} \quad (52)$$

where $M_k \in \mathcal{M}_k(\mathbb{R}^2)$ and $\gamma_2 = \mu_1 + \mu_2 + 1$. The formulas (52), given in [3] for arbitrary dimension, are easily obtained from the relations

$$\tilde{D} \tilde{x}^{2\beta} M_k = 2\beta \tilde{x}^{2\beta-1} M_k, \quad \tilde{D} \tilde{x}^{2\beta+1} M_k = 2(\beta + k + \gamma_2) \tilde{x}^{2\beta} M_k,$$

which follow from the commutation relations

$$[\tilde{D}, \tilde{x}^2] = 2\tilde{x}, \quad \{\tilde{D}, \tilde{x}\} = 2(\tilde{E} + \gamma_2). \quad (53)$$

Similar formulas hold in the one-dimensional case. To present the result, we shall need the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined as [21]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2} \right).$$

The following identity:

$$(x+y)^m P_m^{(\alpha,\beta)} \left(\frac{x-y}{x+y} \right) = \frac{(\alpha+1)_m}{m!} x^m {}_2F_1 \left(\begin{matrix} -m, -m-\beta \\ \alpha+1 \end{matrix} \middle| -\frac{y}{x} \right),$$

will also be needed.

Computing (51) using the definitions (49), (50), the formulas (52) and the above identity, a long but otherwise straightforward calculation shows that the basis spinors have the expression

$$\psi_{k,\pm}^{(N)} = q_{N-k}(x_3, \tilde{x}) m_k(x_2, x_1) \chi_{\pm}, \quad k = 0, \dots, N, \quad (54)$$

where

$$m_k(x_1, x_2) = \mathbf{CK}_{x_2}^{\mu_2} [x_1^k].$$

One has

$$q_{N-k}(x_3, \tilde{x}) = \frac{\beta!}{(\mu_3 + 1/2)_{\beta}} (x_1^2 + x_2^2 + x_3^2)^{\beta} \times \begin{cases} P_{\beta}^{(\mu_3-1/2, k+\mu_1+\mu_2)} \left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2} \right) & N-k=2\beta, \\ -\frac{\sigma_3 x_3 \tilde{x}}{x_1^2+x_2^2+x_3^2} P_{\beta-1}^{(\mu_3+1/2, k+\mu_1+\mu_2+1)} \left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2} \right), \\ \tilde{x} P_{\beta}^{(\mu_3-1/2, k+\mu_1+\mu_2+1)} \left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2} \right), & N-k=2\beta+1, \\ -\sigma_3 x_3 \left(\frac{k+\beta+\mu_1+\mu_2+1}{\beta+\mu_3+1/2} \right) P_{\beta}^{(\mu_3+1/2, k+\mu_1+\mu_2)} \left(\frac{x_1^2+x_2^2-x_3^2}{x_1^2+x_2^2+x_3^2} \right), \end{cases} \quad (55)$$

and

$$m_k(x_2, x_1) = \frac{\beta!}{(\mu_2 + 1/2)_{\beta}} (x_1^2 + x_2^2)^{\beta} \times \begin{cases} P_{\beta}^{(\mu_2-1/2, \mu_1-1/2)} \left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2} \right) - \frac{\sigma_2 x_2 \sigma_1 x_1}{x_1^2+x_2^2} P_{\beta-1}^{(\mu_2+1/2, \mu_1+1/2)} \left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2} \right), & k=2\beta, \\ x_1 P_{\beta}^{(\mu_2-1/2, \mu_1+1/2)} \left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2} \right) - \sigma_2 x_2 \sigma_1 \left(\frac{\beta+\mu_1+1/2}{\beta+\mu_2+1/2} \right) P_{\beta}^{(\mu_2+1/2, \mu_1-1/2)} \left(\frac{x_1^2-x_2^2}{x_1^2+x_2^2} \right), & k=2\beta+1. \end{cases} \quad (56)$$

5.3 Basis spinors and representations of the Bannai-Ito algebra

The basis vectors $\psi_{k,\pm}^{(N)}$ transform irreducibly under the action of the Bannai-Ito algebra. This can be established as follows. By construction, $\psi_{k,\pm}^{(N)} \in \mathcal{M}_N(\mathbb{R}^3)$, and thus (17) gives

$$(\Gamma+1) \psi_{k,\pm}^{(N)} = (N + \mu_1 + \mu_2 + \mu_3 + 1) \psi_{k,\pm}^{(N)}.$$

Hence we have

$$Q \psi_{k,\pm}^{(N)} = ((\Gamma+1)^2 + \mu_1^2 + \mu_2^2 + \mu_3^2 - 1/4) \psi_{k,\pm}^{(N)}, \quad (57)$$

as in (30). The spinors (54) are also eigenvectors of K_3 . To prove this result, one first observes that K_3 can be written as

$$K_3 = -\frac{1}{2}([\tilde{x}, \tilde{D}] + 1)R_1R_2.$$

Since K_3 acts only on the variables (x_1, x_2) and since $[K_3, \tilde{x}] = 0$, one has

$$\begin{aligned} K_3 \psi_{k,\pm}^{(N)} &= K_3 \mathbf{CK}_{x_3}^{\mu_3} \left[\tilde{x}^{N-k} \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] \right] \chi_{\pm} = \mathbf{CK}_{x_3}^{\mu_3} \left[\tilde{x}^{N-k} K_3 \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] \right] \chi_{\pm} \\ &= -\frac{(-1)^k}{2} \mathbf{CK}_{x_3}^{\mu_3} \left[\tilde{x}^{N-k} \left(\tilde{x} \tilde{D} - \tilde{D} \tilde{x} + 1 \right) \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] \right] \chi_{\pm} \\ &= -\frac{(-1)^k}{2} \mathbf{CK}_{x_3}^{\mu_3} \left[\tilde{x}^{N-k} \left(-2(\tilde{E} + \gamma_2) + 1 \right) \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] \right] \chi_{\pm}, \end{aligned}$$

where in the last step the commutation relations (53) were used. Using the properties

$$\tilde{D} \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] = 0, \quad \tilde{E} \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] = k \mathbf{CK}_{x_2}^{\mu_2} [x_1^k], \quad R_1R_2 \mathbf{CK}_{x_2}^{\mu_2} [x_1^k] = (-1)^k \mathbf{CK}_{x_2}^{\mu_2} [x_1^k].$$

one finds that

$$K_3 \psi_{k,\pm}^{(N)} = (-1)^k (k + \mu_1 + \mu_2 + 1/2) \psi_{k,\pm}^{(N)}. \quad (58)$$

Upon combining (57) and (58), it is seen that the spinors (51) satisfy the defining properties of the basis vectors $|N, k\rangle$ for the representations of the Bannai–Ito algebra constructed in section 4. The spinors $\psi_{k,\pm}^{(N)}$ however possess an extra label \pm associated to the eigenvalues of the symmetry operator $Z_3 = \sigma_3 R_3$. Indeed, it is directly verified from the explicit expression (55) and (56) that one has

$$Z_3 \psi_{k,\pm}^{(N)} = \pm (-1)^{N-k} \psi_{k,\pm}^{(N)}.$$

It follows that each of the two independent sets of basis vectors

$$\{\psi_{k,+}^{(N)} \mid k = 0, 1, \dots, N\}, \quad \{\psi_{k,-}^{(N)} \mid k = 0, 1, \dots, N\},$$

supports a unitary $(N+1)$ -dimensional irreducible representation of the Bannai–Ito algebra as constructed in section 4. As a consequence, the space of Dunkl monogenics $\mathcal{M}_N(\mathbb{R}^3)$ of degree N can be expressed as a direct sum of two such representations. Since $\dim \mathcal{M}_N(\mathbb{R}^3) = 2 \times (N+1)$, the dimensions of the spaces match.

5.4 Normalized wavefunctions

The wavefunctions (54) can be presented in a normalized fashion. We define

$$\Psi_{k,\pm}^{(N)}(x_1, x_2, x_3) = \Theta_{N,k}(x_1, x_2, x_3) \Phi_k(x_1, x_2) \chi_{\pm}, \quad (59)$$

with

$$\begin{aligned} \Phi_k(x_1, x_2) &= \sqrt{\frac{\beta! \Gamma(\beta + \mu_1 + \mu_2 + 1)}{2\Gamma(\beta + \mu_1 + 1/2)\Gamma(\beta + \mu_2 + 1/2)}} (x_1^2 + x_2^2)^\beta \\ &\times \begin{cases} P_\beta^{(\mu_2-1/2, \mu_1-1/2)} \left(\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \right) \mathbb{1} & k = 2\beta, \\ + \frac{\sigma_1 x_1 \sigma_2 x_2}{x_1^2 + x_2^2} P_{\beta-1}^{(\mu_2+1/2, \mu_1+1/2)} \left(\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \right), \\ \sqrt{\frac{\beta + \mu_2 + 1/2}{\beta + \mu_1 + 1/2}} x_1 P_\beta^{(\mu_2-1/2, \mu_1+1/2)} \left(\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \right) \mathbb{1} & k = 2\beta + 1, \\ - \sqrt{\frac{\beta + \mu_1 + 1/2}{\beta + \mu_2 + 1/2}} \sigma_2 \sigma_1 x_2 P_\beta^{(\mu_2+1/2, \mu_1-1/2)} \left(\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \right), \end{cases} \end{aligned} \quad (60)$$

and where

$$\Theta_{N,k}(x_1, x_2, x_3) = \sqrt{\frac{\beta! \Gamma(\beta + k + \mu_1 + \mu_2 + \mu_3 + 3/2)}{\Gamma(\beta + \mu_3 + 1/2) \Gamma(\beta + k + \mu_1 + \mu_2 + 1)}} \times \begin{cases} P_{\beta}^{(\mu_3 - 1/2, k + \mu_1 + \mu_2 + 1)} \left(\frac{x_1^2 + x_2^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2} \right) \mathbb{1} & N - k = 2\beta, \\ + \frac{(\sigma_1 x_1 + \sigma_2 x_2) \sigma_3 x_3}{x_1^2 + x_2^2 + x_3^2} P_{\beta-1}^{(\mu_3 + 1/2, k + \mu_1 + \mu_2 + 1)} \left(\frac{x_1^2 + x_2^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2} \right), & \\ \sqrt{\frac{\beta + \mu_3 + 1/2}{\beta + k + \mu_1 + \mu_2 + 1}} (\sigma_1 x_1 + \sigma_2 x_2) P_{\beta}^{(\mu_3 - 1/2, k + \mu_1 + \mu_2 + 1)} \left(\frac{x_1^2 + x_2^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2} \right) & N - k = 2\beta + 1, \\ - \sqrt{\frac{k + \beta + \mu_1 + \mu_2 + 1}{\beta + \mu_3 + 1/2}} \sigma_3 x_3 P_{\beta}^{(\mu_3 + 1/2, k + \mu_1 + \mu_2)} \left(\frac{x_1^2 + x_2^2 - x_3^2}{x_1^2 + x_2^2 + x_3^2} \right), & \end{cases} \quad (61)$$

In (60) and (61), the symbol $\mathbb{1}$ stands for the 2×2 identity operator and $\Gamma(x)$ is the standard Gamma function [1]. Introduce the scalar product

$$\langle \Lambda, \Psi \rangle = \int_{S^2} (\Lambda^\dagger \cdot \Psi) h(x_1, x_2, x_3) dx_1 dx_2 dx_3, \quad (62)$$

where $h(x_1, x_2, x_3)$ is the \mathbb{Z}_2^3 invariant weight function [11]

$$h(x_1, x_2, x_3) = |x_1|^{2\mu_1} |x_2|^{2\mu_2} |x_3|^{2\mu_3}.$$

It is directly verified (see for example [13]) that the spherical Dirac-Dunkl operator Γ and its symmetry operators K_i , Z_i are self-adjoint with respect to the scalar product (62). Upon writing the wavefunctions (59) in the spherical coordinates, it follows from the orthogonality relation of the Jacobi polynomials (see for example [21]) that the wavefunctions (59) satisfy the orthogonality relation

$$\langle \Psi_{k',j}^{(N')}, \Psi_{k,j'}^{(N)} \rangle = \delta_{kk'} \delta_{NN'} \delta_{jj'}.$$

5.5 Role of the Bannai–Ito polynomials

Let us briefly discuss the role played by the Bannai–Ito polynomials in the present picture. It is known that these polynomials arise as overlap coefficients between the respective eigenbases of any pair of generators of the Bannai–Ito algebra in the representations (30) [17, 27]. We introduce the basis $\Upsilon_{s,\pm}^{(N)}$ defined by

$$\Upsilon_{s,\pm}^{(N)} = \tilde{\Theta}_{N,s}(x_2, x_3, x_1) \tilde{\Phi}_s(x_2, x_3) \chi_{\pm}, \quad s = 0, \dots, N, \quad (63)$$

where $\tilde{\Theta}$ and $\tilde{\Phi}$ are obtained from (60) and (61) by applying the permutation $(\mu_1, \mu_2, \mu_3) \rightarrow (\mu_2, \mu_3, \mu_1)$. It is easily seen from (22) and (24) that the wavefunctions (63) satisfy the eigenvalue equations

$$\begin{aligned} (\Gamma + 1) \Upsilon_{s,\pm}^{(N)} &= (N + \mu_1 + \mu_2 + \mu_3 + 1) \Upsilon_{s,\pm}^{(N)}, \\ K_1 \Upsilon_{s,\pm}^{(N)} &= (-1)^s (s + \mu_2 + \mu_3 + 1/2) \Upsilon_{s,\pm}^{(N)}, \\ \sigma_3 R_3 \Upsilon_{s,\pm}^{(N)} &= \pm (-1)^{N-s} \Upsilon_{s,\pm}^{(N)}. \end{aligned}$$

With the scalar product (62), the overlap coefficients between the bases $\Psi_{k,\pm}^{(N)}$ and $\Upsilon_{s,\pm}^{(N)}$ are defined as

$$\langle \Upsilon_{s,q}^{(N)}, \Psi_{k,r}^{(N)} \rangle = W_{s,k;q,r}^{(N)}.$$

The coefficients $W_{s,k;q,r}^{(N)}$ can be expressed in terms of the Bannai–Ito polynomials (see [19]).

6 Conclusion

In this paper, we considered the Dirac–Dunkl operator on the two-sphere associated to the \mathbb{Z}_2^3 Abelian reflection group. We have obtained its symmetries and shown that they generate the Bannai–Ito algebra. We have built the relevant representations of the Bannai–Ito algebra using ladder operators. Finally, using a Cauchy–Kovalevskaja extension theorem, we have constructed the eigenfunctions of the spherical Dirac–Dunkl operator and we have shown that they transform according to irreducible representations of the Bannai–Ito algebra.

It was seen in this paper, the (1) can be considered as a three-parameter deformation of the algebra sl_2 and as such, it can be considered to have rank one. It would of great interest in the future to generalize the Bannai–Ito algebra to arbitrary rank. In that regard, the study of the Dirac–Dunkl operator in n dimensions associated to the \mathbb{Z}_2^n reflection group is interesting.

Acknowledgements

H.D.B. is grateful for the hospitality extended to him by the Centre de recherches mathématiques, where this research was carried. V.X.G. holds an Alexander–Graham–Bell fellowship from the Natural Science and Engineering Research Council of Canada (NSERC). The research of L.V. is supported in part by NSERC.

References

- [1] G. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2001.
- [2] H. De Bie, V. X. Genest, S. Tsujimoto, L. Vinet, and A. Zhedanov. The Bannai–Ito algebra and some applications. *Journal of Physics: Conference Series*, 2015.
- [3] H. De Bie, B. Ørsted, P. Somberg, and V. Souček. Dunkl operators and a family of realizations of $\mathfrak{osp}(1|2)$. *Transactions of the American Mathematical Society*, 364(7):3875–3902, July 2012.
- [4] H. De Bie and N. De Schepper. Clifford–Gegenbauer polynomials related to the Dunkl Dirac operator. *The Bulletin of the Belgian Mathematical Society–Simon Stevin*, 18(2):193–214, May 2011.
- [5] F. Brackx, R. Delanghe, and F. Sommen. *Clifford analysis*. Addison–Wesley, 1982.
- [6] P. Cerejeiras, U. Kähler, and G. Ren. Clifford analysis for finite reflection groups. *Complex Variables and Elliptic Equations*, 51(5-6):487–495, October 2006.
- [7] F. Dai and Y. Xu. *Approximation Theory and Harmonic Analysis on Spheres and Balls*. Springer, 2013.
- [8] C. Dunkl. Reflection groups and orthogonal polynomials on the sphere. *Mathematische Zeitschrift*, 197(13):33–60, 1988.
- [9] C. Dunkl. Differential-difference operators associated to reflection groups. *Transactions of the American Mathematical Society*, 311(13):167–183, January 1989.
- [10] C. Dunkl. Integral kernels with reflection group invariance. *Canadian Journal of Mathematics*, 43(6):1213–1227, December 1991.
- [11] C. Dunkl and Y. Xu. *Orthogonal Polynomials of Several Variables*. Cambridge University Press, 2nd edition, 2014.
- [12] L. Frappat, A. Sciarrino, and P. Sorba. *Dictionary on Lie Algebras and Superalgebras*. Academic Press, 2000.
- [13] V. X. Genest, M. Ismail, L. Vinet, and A. Zhedanov. The Dunkl oscillator in the plane: I. Superintegrability, separated wavefunctions and overlap coefficients. *Journal of Physics A: Mathematical and Theoretical*, 46(14):145201, April 2013.
- [14] V. X. Genest, M. Ismail, L. Vinet, and A. Zhedanov. The Dunkl Oscillator in the Plane II: Representations of the Symmetry Algebra. *Communications in Mathematical Physics*, 329(3):999–1029, August 2014.
- [15] V. X. Genest, L. Vinet, and A. Zhedanov. Bispectrality of the Complementary Bannai–Ito polynomials. *SIGMA*, 9:18, March 2013.
- [16] V. X. Genest, L. Vinet, and A. Zhedanov. The Bannai–Ito algebra and a superintegrable system with reflections on the two-sphere. *Journal of Physics A: Mathematical and Theoretical*, 47(20):205202, May 2014.
- [17] V. X. Genest, L. Vinet, and A. Zhedanov. The Bannai–Ito polynomials as Racah coefficients of the $sl_{-1}(2)$ algebra. *Proceedings of the American Mathematical Society*, 142(5):1545–1560, May 2014.
- [18] V. X. Genest, L. Vinet, and A. Zhedanov. The Dunkl oscillator in three dimensions. *Journal of Physics: Conference Series*, 512(13):012010, 2014.

- [19] V. X. Genest, L. Vinet, and A. Zhedanov. A Laplace-Dunkl equation on S^2 and the Bannai–Ito algebra. *Communications in Mathematical Physics (in Press)*, 2015.
- [20] P. Graczyk, M. Rösler, and M. Yor, editors. *Harmonic and Stochastic Analysis of Dunkl Processes*. Hermann, 2008.
- [21] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q -analogues*. Springer, 2010.
- [22] A. Lesniewski. A remark on the Casimir elements of Lie superalgebras and quantized Lie superalgebras. *Journal of Mathematical Physics*, 36(3):1457–1461, March 1995.
- [23] B. Ørsted, P. Somberg, and V. Souček. The Howe duality for the Dunkl version of the Dirac operator. *Advances in Applied Clifford Algebras*, 19(2):403–415, July 2009.
- [24] M. Rösler. Dunkl Operators: Theory and Applications. In *Orthogonal Polynomials and Special Functions*, volume 1817 of *Lecture Notes in Mathematics*, pages 93–135. Springer, 2003.
- [25] M. Rösler and M. Voit. Markov Processes Related with Dunkl Operators. *Advances in Applied Mathematics*, 21(4), November 1998.
- [26] S. Tsujimoto, L. Vinet, and A. Zhedanov. From $sl_q(2)$ to a parabosonic Hopf algebra. *SIGMA*, 7:93–105, 2011.
- [27] S. Tsujimoto, L. Vinet, and A. Zhedanov. Dunkl shift operators and Bannai–Ito polynomials. *Advances in Mathematics*, 229(4):2123–2158, March 2012.
- [28] L. Vinet and J. F. Van Diejen, editors. *Calogero-Moser-Sutherland models*. Springer, 2000.